

## FANO HYPERSURFACES IN POSITIVE CHARACTERISTIC

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ABSTRACT. We prove that a general Fano hypersurface in a projective space over an algebraically closed field of arbitrary characteristic is separably rationally connected.

## 1. INTRODUCTION

In this paper, we work with varieties over an algebraically closed field  $k$  of arbitrary characteristic.

**Definition 1.1** ([Kol96] IV.3). Let  $X$  be a variety defined over  $k$ .

A variety  $X$  is *rationally connected* if there is a family of irreducible proper rational curves  $g : U \rightarrow Y$  and an evaluation morphism  $u : U \rightarrow X$  such that the morphism  $u^{(2)} : U \times_Y U \rightarrow X \times X$  is dominant.

A variety  $X$  is *separably rationally connected* if there exists a proper rational curve  $f : \mathbb{P}^1 \rightarrow X$  such that the image lies in the smooth locus of  $X$  and the pullback of the tangent sheaf  $f^*TX$  is ample. Such rational curves are called *very free* curves.

We refer to Kollár's book [Kol96] or the work of Kollár-Miyaoka-Mori [KMM92] for the background. If  $X$  is separably rationally connected, then  $X$  is rationally connected. The converse is true when the ground field is of characteristic zero by using the generic smoothness for the dominant map  $u^{(2)}$ . In positive characteristics, the converse statement is open.

In characteristic zero, a very important class of rationally connected varieties are Fano varieties, i.e., smooth varieties with ample anticanonical bundles. In positive characteristic, we only know that they are rationally chain connected.

**Question 1.2** (Kollár). *In arbitrary characteristic, are Fano varieties separably rationally connected?*

The question is open even for Fano hypersurfaces in projective spaces. In this paper, we prove the following theorem.

**Theorem 1.3.** *In arbitrary characteristic, a general Fano hypersurface of degree  $n$  in  $\mathbb{P}_k^n$  contains a minimal very free rational curve of degree  $n$ , i.e., the pullback of the tangent bundle has the splitting type  $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(n-2)}$ .*

**Theorem 1.4.** *In arbitrary characteristic, a general Fano hypersurface in  $\mathbb{P}_k^n$  is separably rationally connected.*

de Jong-Starr [dJS03] proved that every family of separably rationally connected varieties over a curve admits a rational section. Thus using Theorem 1.4, we give another proof of Tsen's theorem.

**Corollary 1.5.** *Every family of Fano hypersurfaces in  $\mathbb{P}^n$  over a curve admits a rational section.*  $\square$

**Acknowledgment.** The author would like to thank his advisor Professor Jason Starr for helpful discussions.

## 2. TYPICAL CURVES AND DEFORMATION THEORY

Let  $n$  be an integer  $\geq 3$ . Let  $X$  be a hypersurface of degree  $n$  in  $\mathbb{P}^n$ . Let  $C$  be a smoothly embedded rational curve of degree  $e$  in  $X$ . We have the normal bundle short exact sequence.

$$0 \longrightarrow TC \longrightarrow TX|_C \longrightarrow \mathcal{N}_{C|X} \longrightarrow 0$$

By adjunction, the degree of  $TX|_C$  is the degree of  $\mathcal{O}_{\mathbb{P}^n}(1)|_C$ . Thus the degree of the normal bundle  $\mathcal{N}_{C|X}$  is  $e - 2$  and the rank is  $n - 2$ .

**Definition 2.1.** Let  $e$  be a positive integer  $\leq n$ . A smoothly embedded rational curve  $C$  of degree  $e$  in  $X$  is *typical*, if the normal bundle is the following:

$$\mathcal{N}_{C|X} \cong \begin{cases} \mathcal{O}^{\oplus(n-3)} \oplus \mathcal{O}(-1), & \text{if } e = 1, \\ \mathcal{O}^{\oplus(n-e)} \oplus \mathcal{O}(1)^{\oplus(e-2)}, & \text{if } e \geq 2. \end{cases}$$

The curve  $C$  is a *typical line* if the degree of  $C$  is one.

Note that when  $e = n$ , typical rational curves of degree  $n$  are very free.

**Lemma 2.2.** *Let  $L$  be a smoothly embedded line in a hypersurface  $X$  of degree  $n$ . Then  $L$  is typical if and only if both of the following conditions hold:*

- (1)  $h^1(C, \mathcal{N}_{L|X}) = 0$ ,
- (2)  $h^1(C, \mathcal{N}_{L|X}(-1)) \leq 1$ .

*Proof.* We may assume that  $\mathcal{N}_{L|X} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{n-2})$ , where  $a_1 \geq \cdots \geq a_{n-2}$ . Condition (1) is equivalent to that  $a_1 \geq \cdots \geq a_{n-2} \geq -1$ . Together with condition (2),  $a_{n-2}$  is either 0 or  $-1$ . When  $a_{n-2} = 0$ ,  $\mathcal{N}_{L|X}$  is semipositive, contradicting with the fact that the degree of  $\mathcal{N}_{C|X}$  is  $-1$ . When  $a_{n-2} = -1$ ,  $\mathcal{N}_{L|X}/\mathcal{O}(a_{n-2})$  is semipositive. Because of the degree of the normal bundle,  $L$  is typical.  $\square$

**Lemma 2.3.** *Let  $C$  be a smoothly embedded rational curve of degree  $e$  in a hypersurface  $X$  of degree  $n$ , where  $2 \leq e \leq n$ . Then  $C$  is typical if and only if both of the following conditions hold:*

- (1)  $h^1(C, \mathcal{N}_{C|X}(-1)) = 0$ ,
- (2)  $h^1(C, \mathcal{N}_{C|X}(-2)) \leq n - e$ .

*Proof.* Recall that the rank of the normal bundle  $\mathcal{N}_{C|X}$  is  $n - 2$  and the degree is  $e - 2$ . We may assume that  $\mathcal{N}_{C|X} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{n-2})$ , where  $a_1 \geq \cdots \geq a_{n-2}$ . Condition (1) is equivalent to that  $a_{n-2} \geq 0$ . Condition (2) implies that at most  $n - e$  of  $a_i$ 's are 0. By degree count,  $C$  is a typical rational curve of degree  $e$ .  $\square$

Typical rational curves in the hypersurface  $X$  are deformation open as very free curves in the following sense.

Let  $H_n$  be the Hilbert scheme of hypersurfaces of degree  $n$  in  $\mathbb{P}^n$ . It is isomorphic to some projective space. Let  $\mathcal{X} \rightarrow H_n$  be the universal hypersurface. The morphism  $\mathcal{X} \rightarrow H_n$  is flat projective and there exists a relative very ample invertible sheaf  $\mathcal{O}_{\mathcal{X}}(1)$  on  $\mathcal{X}$ .

Let  $R_{e,n}$  be the Hilbert scheme parameterizing flat projective families of one-dimensional subschemes in  $\mathcal{X}$  with the Hilbert polynomial  $P(d) = ed + 1$ . By [Kol96] Theorem 1.4,  $R_{e,n}$  is projective over  $H_n$ .

Let  $\mathcal{C}$  be the universal families over  $R_{e,n}$ , denoted by  $\pi : \mathcal{C} \rightarrow R_{e,n}$ . We have the following diagram,

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{i} & R_{e,n} \times_{H_n} \mathcal{X} \\ \pi \downarrow & \swarrow & \\ R_{e,n} & & \end{array}$$

where  $i$  is a closed immersion.

**Proposition 2.4.** *Let  $e$  be a positive integer  $\leq n$ . There exists an open subset in  $R_{e,n}$  parameterizing typical curves of degree  $e$  in hypersurfaces of degree  $n$ .*

*Proof.* Every typical curve of degree  $e$  in a hypersurface of degree  $n$  gives a point in  $R_{e,n}$ . Any small deformation of a smoothly embedded rational curve is still a smoothly embedded rational curve. Thus the proposition follows by Lemma 2.2, Lemma 2.3 and the upper semicontinuity theorem [Har77] III.12.8.  $\square$

**Lemma 2.5.** *There exists an open subset in  $R_{e,n}$  such that for every closed point  $(C, X)$  in the open subset,  $C$  lies in the smooth locus of  $X$ .*

*Proof.* Let  $S \subset \mathcal{X}$  be the relative singular locus in the universal hypersurface.  $S$  is a closed subset of  $\mathcal{X}$ . Since  $\pi$  is proper, the locus  $\pi(i^{-1}(R_{e,n} \times_{H_n} S))$  is a closed subset of  $R_{e,n}$  parametrizing the point  $(C', X')$  such that  $C'$  intersects the singular locus of  $X'$ . Thus the complement  $U$  is open in  $R_{e,n}$  and satisfies the desired property.  $\square$

Let  $L$  be a typical line in a hypersurface  $X$  of degree  $n$  in  $\mathbb{P}^n$ . By definition,  $\mathcal{N}_{L|X} \cong \mathcal{O}^{\oplus(n-3)} \oplus \mathcal{O}(-1)$ . We have a canonically defined *trivial subbundle*  $\mathcal{O}^{\oplus(n-2)}$  of  $\mathcal{N}_{L|X}$ .

**Proposition 2.6.** *Let  $X$  be a hypersurface of degree  $n$  in  $\mathbb{P}^n$ . Let  $L$  and  $M$  be two typical lines in  $X$  intersecting transversally at only one point  $p$ . Assume that the following conditions hold:*

- (1) *the direction  $T_p L$  is not in the trivial subbundle of  $\mathcal{N}_{M|X}$ ;*
- (2) *the direction  $T_p M$  is not in the trivial subbundle of  $\mathcal{N}_{L|X}$ .*

*Then the pair  $(L \cup M, X) \in R_{2,n}$  can be smoothed to a pair  $(C, X')$  where  $C$  is a typical conic in  $X'$ . Furthermore, there exists an open neighborhood of  $(L \cup M, X)$  in which any smoothing of  $(L \cup M, X)$  is a typical conic.*

*Proof.* Let  $D$  be the union of the lines  $L$  and  $M$ . Since  $D$  is a local complete intersection and lies in the smooth locus of  $X$ , the normal bundle  $\mathcal{N}_{D|X}$  is locally free. We have the following short exact sequence.

$$0 \longrightarrow \mathcal{N}_{L|X} \longrightarrow \mathcal{N}_{D|X}|_L \longrightarrow T_p L \otimes T_p M \longrightarrow 0$$

By [GHS03] Lemma 2.6, the locally free sheaf  $\mathcal{N}_{D|X}|_L$  is the sheaf of rational sections of  $\mathcal{N}_{L|X}$  which has at most one pole at the direction of  $T_p M$ . Since  $\mathcal{N}_{L|X} \cong \mathcal{O}^{\oplus(n-3)} \oplus \mathcal{O}(-1)$ , condition (2) implies that  $\mathcal{N}_{D|X}|_L$  is isomorphic to  $\mathcal{O}^{\oplus(n-2)}$ .

By the same argument, condition (1) implies that the sheaf  $\mathcal{N}_{D|X}|_M$  is isomorphic to  $\mathcal{O}^{\oplus(n-2)}$ . Now we have the following short exact sequence.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{N}_{D|X}|_M(-p) & \longrightarrow & \mathcal{N}_{D|X} & \longrightarrow & \mathcal{N}_{D|X}|_L \longrightarrow 0 \\ & & \parallel & & & & \parallel \\ & & \mathcal{O}(-1)^{\oplus(n-2)} & & & & \mathcal{O}^{\oplus(n-2)} \end{array}$$

First we claim that  $D$  can be smoothed. Since  $h^1(D, \mathcal{N}_{D|X}) = 0$ , the pair  $(D, X)$  is unobstructed in  $R_{2,n}$ , cf. [Kol96] I.2. By [Sta09] Lemma 3.17, it suffices to show that the map

$$H^0(D, \mathcal{N}_{D|X}) \rightarrow H^0(L, \mathcal{N}_{D|X}|_L) \rightarrow T_p L \otimes T_p M$$

is surjective. Since  $H^1(M, \mathcal{N}_{D|X}|_M(-p)) = 0$ , the first map is surjective. Since  $H^1(L, \mathcal{N}_{D|X}|_L) = 0$ , the second map is surjective.

Let  $q, r$  be two distinct points on  $L - \{p\}$ . By the long exact sequence associated to the above short exact sequence at  $h^1$ , we get  $h^1(D, \mathcal{N}_{D|X}(-q)) = 0$  and  $h^1(D, \mathcal{N}_{D|X}(-q-r)) = n-2$ .

Now for any smoothing  $(D_t, X_t)$  of  $(D, X)$  over  $T$ , we can specify two distinct points  $p_t$  and  $q_t$  on  $D_t$  which specialize to  $q$  and  $r$  on  $D$ . By Lemma 2.5, after shrinking  $T$ , the conic  $D_t$  lies in the smooth locus of  $X_t$ . Thus  $D_t$  is smoothly embedded. By the upper semicontinuity theorem and Lemma 2.3,  $D_t$  is a typical conic in  $X_t$ .  $\square$

**Definition 2.7.** Let  $X$  be a hypersurface of degree  $n$  in  $\mathbb{P}^n$ . A *typical comb* with  $m$  teeth in  $X$  is a reduced curve in  $X$  with  $m+1$  irreducible components  $C, L_1, \dots, L_m$  satisfying the following conditions:

- (1)  $C$  is a typical conic in  $X$ ;
- (2)  $L_1, \dots, L_m$  are disjoint typical lines in  $X$  and each  $L_i$  intersects  $C$  transversally at  $p_i$ .

The conic  $C$  is called the *handle* of the comb and  $L_i$ 's are called the *teeth*.

**Proposition 2.8.** Let  $X$  be a hypersurface of degree  $n$  in  $\mathbb{P}^n$ . Let  $D = C \cup L_1 \cup \dots \cup L_{n-2}$  be a typical comb with  $n-2$  teeth in  $X$ . Let  $p_i$  be the intersection point  $L_i \cap C$ . Assume that the following conditions hold:

- (1) the direction  $T_{p_i} C$  is not in the trivial subbundle of  $\mathcal{N}_{L_i|X}$ ;
- (2) the directions  $T_{p_i} L_i$  are general in  $\mathcal{N}_{C|X}$  such that the sheaf  $\mathcal{N}_{D|X}|_C$  is isomorphic to  $\mathcal{O}(1)^{\oplus(n-2)}$ .

Then the pair  $(D, X) \in R_{n,n}$  can be smoothed to a pair  $(C', X')$  where  $C'$  is a very free curve in  $X'$ .

*Proof.* The proof is very similar to the proof of Proposition 2.6. Here we only sketch the proof. Condition (1) implies that the sheaf  $\mathcal{N}_{D|X}|_{L_i}$  is isomorphic to  $\mathcal{O}^{\oplus(n-2)}$  for each  $i$ . We have the following short exact sequence.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \sqcup_i \mathcal{N}_{D|X}|_{L_i}(-p) & \longrightarrow & \mathcal{N}_{D|X} & \longrightarrow & \mathcal{N}_{D|X}|_C \longrightarrow 0 \\ & & \parallel & & & & \parallel \\ & & \mathcal{O}(-1)^{\oplus(n-2)} & & & & \mathcal{O}(1)^{\oplus(n-2)} \end{array}$$

Since  $H^1(D, \mathcal{N}_{D|X}) = 0$ ,  $D$  is unobstructed. By diagram chasing, the map  $H^0(D, \mathcal{N}_{D|X}) \rightarrow \sqcup_i T_{p_i} C \otimes T_{p_i} L_i$  is surjective. Thus we can smooth the typical comb  $D$ .

Now we may choose a smoothing  $(D_t, X_t)$  and specify two distinct points  $(q_t, r_t)$  which specialize to two distinct points  $(q, r)$  on  $C - \{p_1, \dots, p_{n-2}\}$ . By the long exact sequence, we know that  $h^1(D, \mathcal{N}_{D|X}(-q - r)) = 0$ . By Lemma 2.5 and the upper semicontinuity theorem, a general smoothing of the pair  $(D, X)$  gives a very free curve in a general hypersurface.  $\square$

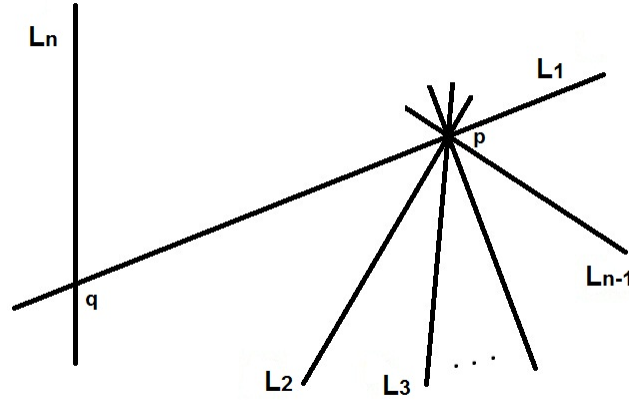
### 3. AN EXAMPLE

In this section, we construct a hypersurface of degree  $n$  in  $\mathbb{P}^n$ , which contains a special configuration of lines. Later we will use this example to produce a very free curve in a general hypersurface.

Let  $n$  be an integer  $\geq 4$ . Let  $[x_0 : \dots : x_n]$  be the homogeneous coordinates for  $\mathbb{P}^n$ . Let  $X$  be a hypersurface of degree  $n$  in the projective space  $\mathbb{P}^n$  defined by the following equation.

$$\begin{array}{ccccccc}
 x_0^{n-1}x_n & +x_1^{n-3}x_n^2x_0 & + (x_1^{n-1} + x_0x_1^{n-2} + \dots + x_0^{n-3}x_1^2)x_2 & + (x_2^{n-1} + x_0x_2^{n-2} + \dots + x_0^{n-3}x_2^2)x_3 & + \dots \\
 & +x_1^{n-4}x_n^3x_3 & + (x_0x_1^{n-2} + \dots + x_0^{n-3}x_1^2)x_3 & + (x_0x_2^{n-2} + \dots + x_0^{n-3}x_2^2)x_4 & + \dots \\
 & \vdots & \vdots & & \\
 & +x_1x_n^{n-2}x_{n-2} & + (x_0^{n-4}x_1^3 + x_0^{n-3}x_1^2)x_{n-2} & + (x_0^{n-4}x_2^3 + x_0^{n-3}x_2^2)x_{n-1} & + \dots \\
 & +x_n^{n-1}x_{n-1} & +x_0^{n-3}x_1^2x_{n-1} & +x_0^{n-3}x_2^2x_1 & + \dots
 \end{array}$$

**Notation 3.1.** Let  $p$  be the point  $[1 : 0 : \dots : 0]$  and  $q$  be the point  $[0 : 1 : 0 : \dots : 0]$ . Let  $L_i$  be the line spanned by  $\{e_0, e_i\}$  for  $i = 1, \dots, n-1$  and  $L_n$  be the line spanned by  $\{e_1, e_n\}$ . It is easy to check that they all lie in the hypersurface  $X$ . Let  $C$  be the union of  $L_1, \dots, L_n$ . The following picture shows the configuration of the points and the lines in the projective space.



- Lemma 3.2.**
- (1) Both  $p$  and  $q$  lie in the smooth locus of  $X$ .
  - (2) The tangent space  $T_p X$  is the hyperplane  $\{x_n = 0\}$ , which is spanned by the lines  $L_1, \dots, L_{n-1}$ .
  - (3) The tangent space of  $T_q X$  is the hyperplane  $\{x_2 = 0\}$ .

*Proof.* By taking the partial derivatives of  $F$ , we have  $\frac{\partial F}{\partial x_i}(p) = 0$  for  $i = 0, \dots, n-1$  and  $\frac{\partial F}{\partial x_n}(p) = 1$ . Similarly, we have  $\frac{\partial F}{\partial x_i}(q) = 0$  for  $i \neq 2$  and  $\frac{\partial F}{\partial x_2}(q) = 1$ .  $\square$

**Lemma 3.3.** *The lines  $L_1, \dots, L_{n-1}$  are in the smooth locus of  $X$ .*

*Proof.* We will prove the case for line  $L_1$ . The remaining cases can be computed directly by the same method. Denote  $L_1 = \{[x_0 : x_1 : 0 : \dots : 0] \in \mathbb{P}^n\}$ . By restricting the partial derivatives of the defining equation of the hypersurface  $X$  on  $L_1$ , we get the following.

$$(3.1) \quad \begin{aligned} \frac{\partial F}{\partial x_2}|_{L_1} &= x_1^{n-1} + x_0 x_1^{n-2} + \dots + x_0^{n-3} x_1^2 \\ \frac{\partial F}{\partial x_3}|_{L_1} &= x_0 x_1^{n-2} + \dots + x_0^{n-3} x_1^2 \\ &\vdots \\ \frac{\partial F}{\partial x_{n-2}}|_{L_1} &= x_0^{n-4} x_1^3 + x_0^{n-3} x_1^2 \\ \frac{\partial F}{\partial x_{n-1}}|_{L_1} &= x_0^{n-3} x_1^2 \\ \frac{\partial F}{\partial x_n}|_{L_1} &= x_0^{n-1} \end{aligned}$$

For points on  $L_1$  with  $x_0 \neq 0$ , we have  $\frac{\partial F}{\partial x_n}|_{L_1} \neq 0$ . At the point  $q$ ,  $\frac{\partial F}{\partial x_2}|_{L_1} \neq 0$ . Hence every point on the line  $L_1$  is a smooth point of  $X$ .  $\square$

**Lemma 3.4.** *The line  $L_n$  is in the smooth locus of  $X$ .*

*Proof.* By restricting the partial derivatives of the defining equation of  $X$  on  $L_n$ , we get the following.

$$(3.2) \quad \begin{aligned} \frac{\partial F}{\partial x_0}|_{L_n} &= x_1^{n-3} x_n^2 \\ \frac{\partial F}{\partial x_3}|_{L_n} &= x_1^{n-4} x_n^3 \\ &\vdots \\ \frac{\partial F}{\partial x_{n-2}}|_{L_n} &= x_1 x_n^{n-2} \\ \frac{\partial F}{\partial x_{n-1}}|_{L_n} &= x_n^{n-1} \\ \frac{\partial F}{\partial x_2}|_{L_n} &= x_1^{n-1} \end{aligned}$$

For points on  $L_n$  with  $x_1 \neq 0$ , we have  $\frac{\partial F}{\partial x_2}|_{L_n} \neq 0$ . For points on  $L_n$  with  $x_n \neq 0$ , we have  $\frac{\partial F}{\partial x_{n-1}}|_{L_n} \neq 0$ . Hence every point on the line  $L_n$  is a smooth point of  $X$ .  $\square$

**Proposition 3.5.** *With the notations as above,  $X$  satisfies the following properties.*

- (1) *The lines  $L_1, \dots, L_n$  are typical in  $X$ .*
- (2) *For  $i = 1, \dots, n-1$ , the trivial subbundle of the normal bundle  $\mathcal{N}_{L_i|X}$  at  $p$  is generated by  $\partial_{\bar{i+1}} - \partial_{\bar{i+2}}, \dots, \partial_{\bar{i+n-3}} - \partial_{\bar{i+n-2}}$ , where  $\bar{j}$  takes value in  $1, \dots, n-1 \bmod n-1$ .*
- (3) *The trivial subbundle of the normal bundle  $\mathcal{N}_{L_1|X}$  at  $q$  is generated by  $\partial_3, \dots, \partial_{n-1}$ .*
- (4) *The trivial subbundle of the normal bundle  $\mathcal{N}_{L_n|X}$  at  $q$  is generated by  $\partial_3, \dots, \partial_{n-1}$ .*

*Proof.* Let  $L$  be a line in  $X$ . We have the following short exact sequences.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{N}_{L|X}(-1) & \longrightarrow & \mathcal{N}_{L|\mathbb{P}^n}(-1) & \longrightarrow & \mathcal{N}_{X|\mathbb{P}^n}|_L(-1) \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & \mathcal{N}_{L|X}(-1) & \longrightarrow & \mathcal{O}_L^{\oplus(n-1)} & \longrightarrow & \mathcal{O}_L(n-1) \longrightarrow 0 \end{array}$$

The associated long exact sequence is the following.

$$H^0(L, \mathcal{N}_{L|X}(-1)) \rightarrow k^n \xrightarrow{\alpha} H^0(L, \mathcal{O}(n-1)) \rightarrow H^1(L, \mathcal{N}_{L|X}(-1)) \rightarrow 0$$

where the map  $\alpha$  sends the natural basis of  $k^n$  to the derivatives of  $F$  with respect to the normal directions of  $L$  in  $\mathbb{P}^n$ . By Lemma 2.2,  $L$  is typical if and only if the image of  $\alpha$  is of codimension one in  $H^0(L, \mathcal{O}(n-1))$ .

When  $L = L_1$ , by (3.1),  $\frac{\partial F}{\partial x_2}|_{L_1}, \dots, \frac{\partial F}{\partial x_n}|_{L_1}$  form a codimensional-one subspace of  $H^0(L_1, \mathcal{O}_{L_1}(n-1))$ . Thus we get that  $H^1(L_1, \mathcal{N}_{L_1|X}(-1))$  is one dimensional, i.e.,  $L_1$  is typical in  $X$ .

By the short exact sequence above,  $\mathcal{N}_{L_1|X}(-1)$  is a subbundle of the trivial bundle  $\mathcal{O}_{L_1}^{\oplus(n-1)}$  which maps to 0 in  $\mathcal{O}_{L_1}(n-1)$ . Let  $\partial_2, \dots, \partial_n$  be the generators of  $\mathcal{O}_{L_1}^{\oplus(n-1)}$ . We get  $\mathcal{N}_{L_1|X}(-1)$  is generated by  $x_0(\partial_2 - \partial_3) - x_1(\partial_3 - \partial_4), \dots, x_0(\partial_{n-2} - \partial_{n-1}) - x_1\partial_{n-1}, x_0^2\partial_{n-1} - x_1^2\partial_n$  as an  $\mathcal{O}_{L_1}$ -module. If we restrict the bundle at  $p$  and  $q$ , we get (2) and (3) for  $L_1$ .

When  $L = L_2, \dots, L_{n-1}$ , we can prove in a similar way. When  $L = L_n$ , (4) follows from the same computation as above by applying (3.2).  $\square$

With the description of the trivial subbundles of the normal bundles of lines in  $X$  as above, we get the following corollaries.

**Corollary 3.6.** *We have the following statements.*

- (1) *The lines  $L_1$  and  $L_n$  are typical in  $X$ .*
- (2) *The direction  $T_q L_1$  is not in the trivial subbundle of  $\mathcal{N}_{L_n|X}$ .*
- (3) *The direction  $T_q L_n$  is not in the trivial subbundle of  $\mathcal{N}_{L_1|X}$ .*  $\square$

**Corollary 3.7.** *We have the following statements.*

- (1) *The lines  $L_2, \dots, L_{n-2}$  are typical in  $X$ .*
- (2) *The direction  $T_p L_1$  is not in the trivial subbundle of  $\mathcal{N}_{L_i|X}$  for  $2 \leq i \leq n-1$ .*
- (3) *The directions  $T_p L_2, \dots, T_p L_{n-1}$  span the normal bundle  $\mathcal{N}_{L_1|X}$  at  $p$ .*  $\square$

#### 4. PROOF OF THE MAIN THEOREM

**Lemma 4.1** ([Har92] Ex 13.8). *Let  $C$  be the union of  $n$  lines  $L_1, \dots, L_n$  in  $\mathbb{P}^n$  as in Notation 3.1. Then the Hilbert polynomial of  $C$  is  $P(d) = nd + 1$ . In particular, the arithmetic genus of  $C$  is 0.*

*Proof.* This can be computed directly. For any  $d > 0$ , when  $i = 1, \dots, n-1$ , the homogeneous polynomials of degree  $d$  that do not vanish on  $L_i$  are generated by  $\{x_0^d, x_0^{d-1}x_i, \dots, x_i^d\}$ . The homogeneous polynomials of degree  $d$  that do not vanish on  $L_n$  are generated by  $\{x_1^d, x_1^{d-1}x_i, \dots, x_n^d\}$ . Thus when  $d \gg 0$ ,  $P(d) = H^0(C, \mathcal{O}_C(d)) = nd + 1$ .  $\square$

The curve  $C$  is an example of curves with rational  $n$ -fold point, cf. [CCC11] 3.7. The following lemma is an analogue of [CCC11] Lemma 3.8.

**Lemma 4.2.** *With the same notations as in Lemma 4.1, the following properties hold for  $C$  for every positive integer  $d$ :*

- (1)  $h^0(C, \mathcal{O}_C(d)) = nd + 1$  and  $h^1(C, \mathcal{O}_C(d)) = 0$ .
- (2)  $h^1(C, \mathcal{I}_C(d)) = 0$ .
- (3)  $h^0(C, \mathcal{I}_C(d)) = h^0(\mathbb{P}^n, \mathcal{O}(d)) - nd - 1$ .

*Proof.* By Riemann-Roch and Lemma 4.1, we have

$$h^0(C, \mathcal{O}_C(d)) \geq \chi(\mathcal{O}_C(d)) = nd + 1 - p_g(C) = nd + 1.$$

On the other hand, every global section of  $\mathcal{O}_C(d)$  is obtained by gluing global sections on each component, which imposes at least  $n - 1$  linear conditions. Since we have  $h^0(L_i, \mathcal{O}(d)) = d + 1$  for every  $i$ ,

$$h^0(\mathcal{O}_C(d)) \leq n(d + 1) - (n - 1) = nd + 1.$$

Therefore,  $h^0(C, \mathcal{O}_C(d)) = nd + 1$  for every positive integer  $d$ . Since the image of  $r : H^0(\mathbb{P}^n, \mathcal{O}(d)) \rightarrow H^0(C, \mathcal{O}_C(d))$  has dimension  $nd + 1$  as in Lemma 4.1, the map  $r$  is surjective. The lemma follows by considering the long exact sequence associated to the ideal sheaf of  $C$ .  $\square$

**Construction 4.3.** Let  $C$  be the union of  $n$  lines  $L_1, \dots, L_n$  in  $\mathbb{P}^n$  as in Notation 3.1. If we consider  $L_1 \cup L_n$  as a conic in  $\mathbb{P}^n$ , there exists a smooth affine pointed curve  $(T, 0)$  and a smoothing  $D' \rightarrow (T, 0)$  satisfying the following conditions:

- (1) The special fiber  $D'_0$  is  $L_1 \cup L_n$ ;
- (2) For any  $t \in T - \{0\}$ ,  $D'_t$  is a smooth conic contained in the plane spanned by  $L_1$  and  $L_n$ .

We can choose  $n - 2$  sections  $s_i : (T, 0) \rightarrow D'$  for  $i = 1, \dots, n - 2$  such that  $s_i(0) = p$  for all  $i$ 's and for  $t \in T - \{0\}$ ,  $s_i(t)$ 's are all distinct on  $D'_t$ .

For any  $s_i(t)$ , there exists a unique line  $L_{i+1}(t)$  through  $s_i(t)$  parallel to  $L_{i+1}$ . After gluing the families of lines  $L_{i+1}(t)$  on  $D'_t$  at  $s_i(t)$  for all  $i$ 's, we get a family of reducible curves  $\pi : D \rightarrow (T, 0)$  satisfying the following conditions:

- (1) The special fiber  $D_0$  is  $C$  as constructed in 3.1.
- (2) For any  $t \in T - \{0\}$ ,  $D_t$  is a comb with the handle  $D'_t$  and with the teeth lines.

The family  $\pi : D \rightarrow (T, 0)$  is flat by Lemma 4.1. We have the following diagram.

$$\begin{array}{ccccc} D_0 = C & \longrightarrow & D & \xrightarrow{i} & \mathbb{P}_T^n \\ \downarrow & & \downarrow \pi & \swarrow \pi & \\ 0 & \longrightarrow & (T, 0) & & \end{array}$$

**Lemma 4.4.** Let  $\mathcal{I}_D$  be the ideal sheaves of  $D$  in  $\mathbb{P}_T^n$ . The sheaf  $\pi_* \mathcal{I}_D(d)$  is locally free on  $T$  for any  $d > 0$ .

*Proof.* By the cohomology and base change theorem [Har77] III.12.9, it suffices to show that  $h^0(\mathbb{P}_t^n, \mathcal{I}_{D_t}(d))$  is constant. By upper semicontinuity and Lemma 4.2, we have  $h^0(\mathbb{P}_t^n, \mathcal{I}_{D_t}(d)) \leq h^0(\mathbb{P}^n, \mathcal{O}(d)) - nd - 1$ . On the other hand, for any  $t \in T - \{0\}$ , the curve  $D_t$  is a local complete intersection and  $\mathcal{O}_{D_t}(d)$  is a positive bundle on  $D_t$ . Thus we have  $h^1(D_t, \mathcal{O}_{D_t}(d)) = 0$  and  $h^0(D_t, \mathcal{O}_{D_t}(d)) = nd + 1$ . Consider the following exact sequence.

$$0 \longrightarrow H^0(\mathbb{P}_t^n, \mathcal{I}_{D_t}(d)) \longrightarrow H^0(\mathbb{P}_t^n, \mathcal{O}(d)) \longrightarrow H^0(D_t, \mathcal{O}_{D_t}(d))$$

We get  $h^0(\mathbb{P}_t^n, \mathcal{I}_{D_t}(d)) \geq h^0(\mathbb{P}^n, \mathcal{O}(d)) - nd - 1$ .  $\square$

*Proof of Theorem 1.3.* The theorem is trivial for  $n = 2, 3$ . We can assume that  $n \geq 4$ . By [Kol96] IV.3.11 and Lemma 2.5, it suffices to produce one very free curve in a hypersurface of degree  $n$ . By Lemma 4.4, after shrinking  $T$ , hypersurfaces of



degree  $n$  containing  $D_t$  in  $\mathbb{P}_T^n$  form a trivial projective bundle over  $(T, 0)$ . Thus the family  $\pi : D \rightarrow (T, 0)$  admits a lifting to a flat family of pairs  $\pi : (D, \mathcal{X}_T) \rightarrow (T, 0)$  in  $R_{n,n}$  such that the special fiber  $(D_0, \mathcal{X}_0)$  is  $(C, X)$  which is constructed in Section 3.

$$\begin{array}{ccccc} D & \xrightarrow{i} & \mathcal{X}_T & \longrightarrow & \mathbb{P}_T^n \\ \pi \downarrow & & \swarrow & \searrow \pi & \\ & & (T, 0) & & \end{array}$$

All the following steps of the proof requires to shrink  $T$  if necessary. By Proposition 2.6 and Corollary 3.6, we may assume that the handle  $D'_t$  is a typical conic in  $\mathcal{X}_t$  for every  $t \in T - \{0\}$ . By Proposition 2.4 and Corollary 3.7 (1), all the teeth of the comb  $D_t$  are typical. Thus for every  $t \in T - \{0\}$ , we get a typical comb  $D_t$  as in Definition 2.7. Now the theorem follows if we verify the two conditions in Proposition 2.3. Since they are open conditions, it suffices to check on the special fiber  $(C, X)$ , which is proved in Corollary 3.7.  $\square$

*Proof of Theorem 1.4.* By [Kol96] IV.3.11 and Lemma 2.5, it suffices to produce one very free curve in a hypersurface of degree  $d$ . Let  $Y$  be a general smooth Fano hypersurface of degree  $d$  in  $\mathbb{P}^n$ . When  $d = n$ , this is proved in Theorem 1.3. When  $d < n$ , we may choose a general linear subspace  $L$  of dimension  $d$  such that  $Y \cap L$  is smooth and contains a very free curve  $f : \mathbb{P}^1 \rightarrow Y \cap L$  by Theorem 1.3. By the normal bundle exact sequence,

$$0 \longrightarrow T(Y \cap L) \longrightarrow TY \longrightarrow \mathcal{N}_{Y \cap L|Y} \longrightarrow 0$$

the sheaf  $f^*T(Y \cap L)$  is positive and the sheaf  $\mathcal{N}_{Y \cap L|Y}$  is isomorphic to  $\mathcal{N}_{L|\mathbb{P}^n}$ , which is  $\mathcal{O}(1)^{\oplus(n-d)}$ . Therefore the pullback bundle  $f^*TY$  is positive. Thus  $f : \mathbb{P}^1 \rightarrow Y \cap L \rightarrow Y$  is a very free curve in  $Y$ .  $\square$

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